Aspects of Singularity Theory

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VENIA LEGENDI

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Duco van Straten

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Preface

This HABILITATIONSSCHRIFT consists of seven papers which were written during the last five years. The first two were written together with T. de Jong, and the fifth with J. Montaldi. I take the opportunity here to thank them for many years of friendship and fruitful collaboration. Of course, also many thanks to all my colleagues and friends in Kaiserslautern, who were always willing to help me.

The papers form a unity in the sense that they all are about certain aspects of *the theory of singularities*, and more specifically, about *deformations* of these. Hopefully, the choice made here also will give some idea of the many aspects of Singularity Theory.

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Introduction

We will understand by the term *singularity* just a *germ of a complex analytic space* or an appropriate *representative* thereof. It has become a custom in singularity theory to use symbols such as X to denote a singularity, and $\mathcal{O}_X$ for their structure ring, rather than the germ notation (X,p), $\mathcal{O}_{(X,p)}$. As most things in singularity theory "are local", this usually does not cause any problems. It should always be clear from the context what is intended.

The first thing one does when one comes across a singularity is to ask about the most basic invariants, its *dimension* $\dim(X)$ (that is, the smallest n such that there is a finite map $X \longrightarrow \mathbf{C}^n$), its *embedding dimension* $\mathrm{embdim}(X)$ (that is, the smallest m such that there is an embedding $X \longrightarrow \mathbf{C}^m$), and then determine its *singular locus* $\Sigma \subset X$ (that is, those points q where $\dim(X,q) < \mathrm{embdim}(X,q)$). If $\Sigma = \{p\}$ we say that X has an *isolated singular point*, or is an *isolated singularity.*

In this Habilitationsschrift we are concerned with various aspects of *deformations of singularities.* Recall that a deformation of X over S is a cartesian diagram

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X} \\ \downarrow & & \pi\downarrow \\ \{0\} & \hookrightarrow & S \end{array}$$

where π is a *flat* map. Here S is also a germ of an analytic space. After taking appropriate representatives for $\mathcal{X}$ and S, we can speak about the fibres $X_s := \pi^{-1}(s)$, $s \in S$. Flatness is a technical condition that guarantees that these fibres form a "sensible" family. For instance, the dimensions are all the same, but it is a fundamental problem to understand the exact relation between X and X_s.

If for generic $s \in S$ the fibre X_s is *smooth*, then we say that the deformation is a *smoothing* of X, and will call such a nearby smooth fibre a *Milnor fibre* of X. Although X is contractible of dimension n, this nearby fibre X_s will in general be some complex n–dimensional Stein manifold with a non–trivial topology. The smoothing of the A_1–singularity $\{(x,y,z)|x^2+y^2-z^2=0\}$ is archetypical:

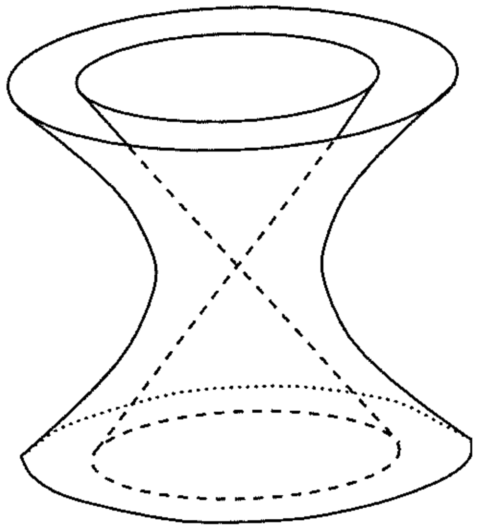

Figure 1: Smoothing of the A_1–singularity

In the above real picture, the Milnor fibre has the homotopy type of the 1–sphere, but in the complex domain it is a 2–sphere. If S is irreducible, then the complement of the set of points where X_s is not smooth, is connected, and hence all the fibres X_s will be diffeomorphic. But if S has more components, then a priori there is the possibility of essentially different topology over the different components. This leads to the following

Problem : *" Given X, how many different X_s are possible?"*

A first step in a systematic study of this problem is the following

Theorem : If X is an isolated singularity, then it has a *semi–universal deformation:*

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X}_{\mathcal{B}} \\ \downarrow & & \pi\downarrow \\ \{0\} & \hookrightarrow & \mathcal{B}. \end{array}$$

Versality means that any given deformation $\mathcal{X} \longrightarrow S$ is *induced* from this one by pull–back via some map $j : S \longrightarrow \mathcal{B}$, that is one has $\mathcal{X} = \mathcal{X}_{\mathcal{B}} \times_{\mathcal{B}} S$. So in some sense $\mathcal{B}$ contains all the deformations of X. The *semi-uni* refers to the fact that by taking $\mathcal{B}$ versal of minimal dimension, the classifying map j is unique on the tangent level, although j itself is *not* unique. For short we will refer to $\mathcal{B}$ as the *base space of* X. It is uniquely determined by X, although only up to non-unique isomorphism.

A version of this theorem, under the condition that the obstruction group (see below) $T^2 = 0$, was given by Tjurina [Tj1] (in these cases $\mathcal{B}$ is always smooth); the general result was obtained by Grauert [Gra]. A formal version of the theorem was known earlier through the work of Schlessinger [Schl1]. Now, in general, there is not much that can be said about $\mathcal{B}$: it could be any sort of analytic space. But one thing is clear from this: only finitely many different fibres X_s can occur. For $\mathcal{B}$ has *finitely many irreducible components* and over each component one has, at most, one type of smooth fibre occurring. If, over some component, smoothing occurs, we call it a *smoothing component of* X. Of course, it might happen that over different components of $\mathcal{B}$ one finds diffeomorphic fibres. In fact, this always happens for (reduced) curve singularities, that is $\dim(X) = 1$. The reason for this is that the topological type of a complex one–dimensional Stein space is determined by the rank of its first homology group. In fact one has the formula (see [B-G], (1.2.1)):

$$\mu := \text{rank } H_1(X_s) = 2\delta(X) - r(X) + 1.$$

Here $\delta(X)$ is the so-called δ-invariant of X, and $r(X)$ is the number of branches of X. Although not all curve singularities have a smoothing ([Pi1], [Gre1]), we see that when they have, the topology of the smooth fibre is determined by X alone.

For surface singularities (that is $\dim(X) = 2$), the situation is very different. H. Pinkham [Pi1] has found the first example of a singularity having more than one, to wit two, different components in its base space. Both components are smoothing components, and the topology of the fibres over the two components is very different. The singularity X can be described as follows: Take $Y = \mathbf{C}^2$, let the cyclic group $G = \mathbf{Z}/4$ act via $(x,y) \mapsto (\sqrt{-1}.x, \sqrt{-1}.y)$, and let X be the quotient space of Y with respect to this group action. The affine coordinate ring of X can be identified with the ring of G-invariant polynomials in x and y, that is $\mathbf{C}[x^4, x^3y, x^2y^2, xy^3, y^4]$. From this one sees that X might alternatively be defined as the projective cone

$$\mathrm{Cone}(C_4) \subset \mathbf{C}^5$$

over the rational normal curve $C_4 = \mathrm{Image}\,(\mathbf{P}^1 \overset{|\mathcal{O}(4)|}{\hookrightarrow} \mathbf{P}^4)$ of degree four. The base space $\mathcal{B}$ turns out to be isomorphic to $\{(e, a_1, a_2, a_3) \in \mathbf{C}^4 | \quad e.a_i = 0 \text{ for } i = 1, 2, 3\}$. We can represent this space schematically by the following picture:

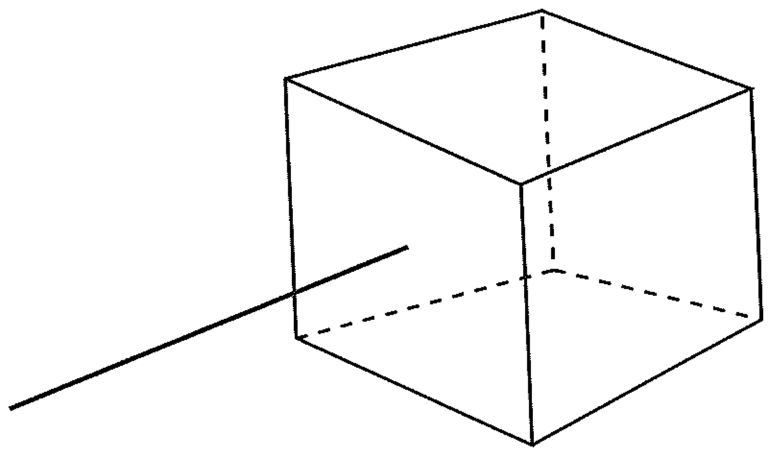

Figure 2: Base space of the Pinkham example

So there are two components, one of dimension one, and one of dimension three, intersecting in the origin. The fibre over the origin is our given X; all other fibres are smooth. The homology groups over the two components are as follows:

big component :	$H_1(X_s) = 0$	, $H_2(X_s) = \mathbf{Z}$
small component :	$H_1(X_s) = \mathbf{Z}/2$	, $H_2(X_s) = 0$

The geometrical reason for the occurrence of these two components is that the curve $C_4 \subset \mathbf{P}^4$ occurs as hyperplane section of *two different surfaces of degree 4* in $\mathbf{P}^4$: the Veronese surface and a scroll.
However, the cones

$$X_n = \mathrm{Cone}(C_n), \quad C_n := \mathrm{Image}\,(\mathbf{P}^1 \overset{|\mathcal{O}(n)|}{\hookrightarrow} \mathbf{P}^n)\,,$$

for $n \neq 4$ have only *one* (smoothing) component. This can be established by a direct calculation which, although not very complicated, somehow fails to *explain* these simple facts. Furthermore, a result of Schlessinger [Schl2] shows that higher dimensional quotient singularities (with singular locus of codimension ≥ 3) are *rigid*, that is have $\mathcal{B} = point$. This shows that something interesting is going on for (normal) surface singularities. In 1981 the paper "Smoothings of normal Surface Singularities", written by J. Wahl, appeared ([Wa6]). Here he discovered the main features of smoothing components of surface singularities. Some of his results were conjectures only, but by the work of Greuel, Looijenga and Steenbrink all his conjectures are now theorems. Among their results we mention:

1. The first betti number $b_1(X_s)$ of the Milnor fibre over any smoothing component is zero ([G-S]). So a Milnor fibre has two interesting homology groups, H_1, a torsion group, and H_2, a free abelian group, of rank μ, the so-called Milnor-number.

2. The difference of 2μ and the dimension of the smoothing component depends only on X, and not on the choice of the smoothing component ([G-L]). In particular, components in the base space of a normal surface singularity can differ in dimension only by *even* numbers. If X is *Gorenstein*, then in fact μ is an invariant of X ([Laufl], [Stee2]) and so all components have the same dimension.

As Pinkham's example shows, it is interesting to study base spaces of even the simplest surface singularities. A natural class is formed by the so-called *rational surface singularities*, introduced by M. Artin in 1966 [Art1]. This broad class, that includes the quotient singularities $\mathbf{C}^2/G$, is defined in terms of a *resolution* $\rho : Y \longrightarrow X$ of X by the condition $R^1\rho_*(\mathcal{O}_Y) = H^1(\mathcal{O}_Y) = 0$. The *exceptional divisor* $E = \rho^{-1}(0)$ consists of a union of $\mathbf{P}^1$'s, intersecting in the pattern of a tree Γ. The condition of rationality can be checked by looking at this resolution graph Γ, where each curve is labelled with its self-intersection in Y. A rational surface singularity X of multiplicity m has embedding dimension $m+1$; those of multiplicity 2 are the well-known **A–D–E**-singularities in three-space ([DuV], [Dur], [Gre2]). As hypersurfaces, these have a smooth bases space $\mathcal{B}$. The rational triple points reside in $\mathbf{C}^4$, and as such are determinantal, giving rise again to a smooth $\mathcal{B}$ [Tj2].

As rationality is defined in terms of a resolution, it seems natural to study the deformation theory also from the resolution. E. Brieskorn ([Bri1], [Bri2], [Bri5] made the beautiful discovery that all rational double points have *simultaneous resolution after base change*. Another way of saying this is that the semi-universal deformation of the resolution Y, which is a smooth space $\mathcal{B}_Y$, maps finitely and surjective to the base space $\mathcal{B}_X$ of X. Furthermore, the resulting map $\mathcal{B}_Y \longrightarrow \mathcal{B}_X$ is a Galois-covering and can be naturally identified with the quotient map by the corresponding Weyl group. These results were extended to more general cases by M. Artin in [Art2] and J. Wahl [Wa3]. It was found that the deformations of a resolution form a smooth space, and all these can be blown down to give deformations of the singularity X and, moreover, the image is always a *component* of $\mathcal{B}$, the so called Artin–component. This explains all deformations in the case that the multiplicity is two or three, but Pinkham's example shows that there are other types of deformations as soon as the multiplicity exceeds three.
The question arises how to determine the base space $\mathcal{B}$ of a given singularity X. In principle, there is a nice theory for doing so. For each singularity X there exists a complex $\mathbf{L}^\bullet_X$ of $\mathcal{O}_X$-modules, called the *cotangent complex*, [L-S]. Loosely speaking, this $\mathbf{L}^\bullet_X$ can be seen as a derived version of the module of Kähler differentials; in fact one has

$$H_0(\mathbf{L}^\bullet_X) = \Omega^1_X.$$

One uses the notation $T^k_X := H^k(Hom(\mathbf{L}^\bullet_X, \mathcal{O}_X))$. There is extensive, but not always easy-to-read literature on the cotangent complex ([An], [Bi], [Buc], [Fl], [Il], [Laud], [Pa]). Its relevance for the deformation theory is explained by the following properties:

$$\begin{aligned} T^0_X &= \text{vector fields on } X \\ T^1_X &= \text{first order deformations of } X \\ T^2_X &= \text{obstruction space of } X. \end{aligned}$$

Here a first order deformation is a deformation of X over the double point $Spec(\mathbb{C}[\epsilon]/(\epsilon)^2)$; the set of all these forms in a natural way a $\mathbb{C}$-vector space. In the case that this space is finite dimensional there exists a semi-universal family for X. The Zariski tangent space to the base $\mathcal{B}$ is just T^1_X. Given a first order deformation of X, we might try to extend it to a deformation to second order, say over $Spec(\mathbb{C}[\epsilon]/(\epsilon)^3)$. This is possible if a certain obstruction in

T^2_X vanishes. Then one lifts the family to second order, and tries to extend to third order, etc. All obstructions encountered in this process are in T^2_X. As a result, one can construct a semi-universal deformation of X over a subscheme of the *space* T^1_X defined as the zero fibre of some analytic map, the *obstruction map*:

$$Ob : T^1_X \longrightarrow T^2_X, \quad \mathcal{B} := Ob^{-1}(0).$$

We see that in this representation the elements of (the dual of) T^2_X can be seen as giving *equations* for the space $\mathcal{B}$ inside T^1_X. So the theory of the cotangent complex suggests that one should take the following steps to obtain $\mathcal{B}$:
1) Determine the space of first order deformations T^1_X.
2) Determine the obstructions space T^2_X.
3) Determine the obstruction map $Ob : T^1_X \longrightarrow T^2_X$
4) Now $\mathcal{B} = Ob^{-1}(0)$. Determine the components of $\mathcal{B}$ and study its properties.
The truth is, that only in very special cases can one really determine $\mathcal{B}$ in this way. A main point is the non–uniqueness of the obstruction map Ob. Even for a rational singularity, defined by some resolution graph, it is in general impossible to determine T^1_X from the resolution data, although some partial results were obtained in [B-K]. This seems to frustrate the above program already at its first step.

Around 1988 several breakthroughs were achieved. First, there was the work of J. Arndt on cyclic quotients [Arn]. He was able to give a more or less explicit set of equations defining $\mathcal{B}$ for a cyclic quotient in terms of the dual resolution graph, by generalizing the equations describing the deformations over the Artin–component that were obtained earlier by Riemenschneider ([Rie2]). He found a very simple formula for the dimension of T^2 (from now on we assume $m \neq 2$):

$$\dim \ T^2 = (m-1)(m-3)$$

where m is the multiplicity of the cyclic quotient. (This was also discovered around the same time by J. Christophersen.) Moreover, he was able to *conjecture*, on the basis of these equations, an upperbound for the number of components of $\mathcal{B}$.
Then, J. Kollár and N. Shepherd-Barron [K-S] applied ideas from the theory

of minimal models of threefolds to smoothing components of surface singularities. Among other things, they showed that all smoothing components of a cyclic quotient come from so-called *P-resolutions*, which are *partial* resolutions $\pi : Y \longrightarrow X$, which have nef relative canonical bundel, and where the space Y has so-called qG-singularities. A qG-singularity is one for which the canonical cover has an equivariant smoothing. In this picture the components arise by blowing down a deformation of the partial resolution that locally induces this qG-smoothing. Pinkham's example is itself a qG-singularity: the small component comes from the fact that the canonical two-fold cover is the ordinary double point $\mathbf{A}_1$. This has a one-dimensional smoothing (this is figure 1!), over which the group action lifts. This nicely "explains" the occurence of $H_1(X_s) = \mathbf{Z}/2$. The idea of using covers to obtain components was used earlier by J. Wahl [Wa5]. Conjecturally, something similar happens for any rational surface singularity ("Kollár's conjectures").

Also around that time, T. de Jong and I started working in a different direction. Inspired by the idea of *series* of isolated hypersurface singularities, D. Siersma started studying so-called isolated line singularities [Si1] from a topological point of view. R. Pellikaan considered hypersurfaces with general one-dimensional singular locus Σ. The most precise results were obtained in the case where Σ was assumed to be a complete intersection, but [Pe3] contained also one tempting example of a hypersurface in $\mathbf{C}^3$, with the union of the three coordinate axes as singular locus. It can be defined as follows. Let

$$F(x, y, z; a, b, c, \mu) := X^2 + Y^2 + Z^2 + 2\lambda.(XY + YZ + ZX) + 2\mu xyz$$

where λ is a fixed complex number $\lambda^2 \neq 1$ and

$$\begin{aligned} X &:= (y-b)(z+c) + 4bc \\ Y &:= (z-c)(x+a) + 4ac \\ Z &:= (x-a)(y+b) + 4ab. \end{aligned}$$

Here a, b, c, μ are parameters. Consider the surface

$$X(a, b, c, \mu) = \{(x, y, z) | F(x, y, z; a, b, c, \mu) = 0\}.$$

If $a = b = c = 0$ and $\mu$is arbitrary then the surface still has the coordinate axes as singular locus, but if $\mu = 0$ and $a, b, c$are arbitrary then these

coordinate axes undergo a flat deformation. In fact, the a, b, c–parameters define a semi–universal deformation of the coordinate axes and for generic a, b, c this becomes a smooth curve. For all other values of the parameters, however, the singular locus of the surface is smaller.

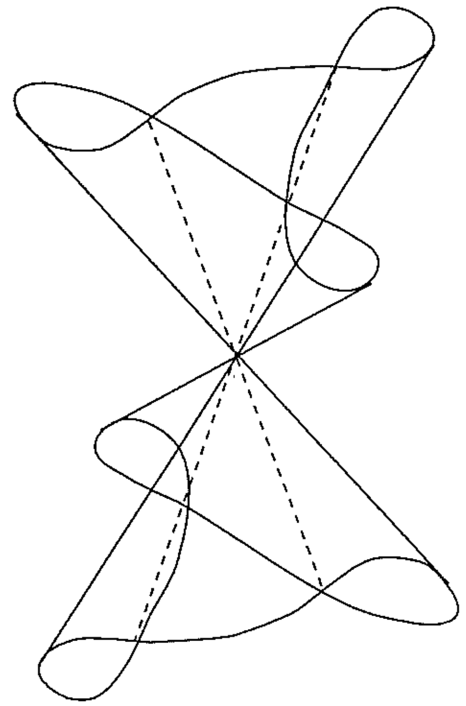

Figure 3: Pellikaan's example

Note that the parameter space is exactly equal to the space of figure 2, the base space of Pinkham's example. This is no coincidence; the *normalization* of the surface $X(0, 0, 0, 0)$ actually **is** Pinkham's example. So in fact we are looking here at a *projection* to $\mathbf{C}^3$ of this example. From this idea we developed the theory of *admissible deformations* $Def(\Sigma, X)$ of hypersurfaces, which consists of deformations of X, together with its singular locus Σ, [J-S1]. It was shown in [J-S2] that the base space of the semi–universal admissible deformation is the same as the base space of the normalization, *up to a smooth factor*.

The big advantage of working with $Def(\Sigma, X)$ rather then directly with the deformations of the normalization, is that now we have a *space curve* Σ and a single *function* f to deal with. Moreover, of f only the *class* $[f]$ in the Artinian $\mathcal{O}_\Sigma$–module $I^{(2)}/I^2$ (where I is the ideal of Σ) is relevant if we want to study the base space up to a smooth factor. This makes the study of space curves very important for the deformation theory of normal surfaces. Together with this principle of I^2 – *equivalence* of singularities, the so–called *projection method* is a very powerful tool for comparing base spaces of geometrically different spaces. In the first paper of this Habilitationsschrift we applied this method to give a description of the base spaces of *all* rational surface singularities of multiplicity four. The result turns out to be surprisingly simple: every rational surface singularity of multiplicity four has a so–called "n-invariant", (called the virtual number of quadruple points in [Stev3]), which can be defined as the number of times one has to blow up X to obtain only singularities of multiplicity lower than four. There exists a universal space $\mathbf{B}(n)$ such that one has:

$$\mathcal{B} = \mathbf{B}(n) \times S$$

where S is a smooth germ. The space $\mathbf{B}(n)$ has $(n+1)$ components; the space $\mathbf{B}(1)$ is just the base space of Pinkham's example, figure 2.
The projection method has also been applied with succes by J. Stevens [Stev4] and S. Brome [Bro] to obtain information about base spaces of certain other types of singularities.
Another important development was started by J. Christophersen. His attack was on the obstruction space T^2, and with K. Behnke, he discovered the beautiful simple fact that the number of generators of the obstruction space T^2 of a rational surface singularity is always $(m-1)(m-3)$. Their idea uses generic hyperplane sections and general properties of the cotangent complex, and is an extension of ideas of Greuel-Looijenga [G-L] and R. Buchweitz [Buc]. In any case, from this work it appeared that the obstruction space T^2 is easier to understand than T^1 !
The combination of this work with what was known for quadruple points prompted K. Behnke, J. Christophersen and T. de Jong to conjecture a formula for the actual dimension of T^2 of a rational surface singularity:

$$\dim\ T^2_X = (m-1)(m-3) + \dim\ T^2_{\widehat{X}}$$

where $\widehat{X}$ is the blow–up of X at the origin. On $\widehat{X}$ one finds a finite number of rational singularities, whose resolution graphs can be obtained easily from

the resolution graph of X by a theorem of Tjurina ([Tj2]), so this formula would enable computing $\dim T^2$ for a rational surface singularity from the resolution graph.
Unfortunately, the formula is not true in general, but it turns out to be true in the case that X is a rational surface singularity *with a reduced fundamental cycle*. Another way of formulating this condition is saying that the general hyperplane section consists of the m coordinate lines in $\mathbf{C}^m$. This broad class includes the cyclic quotients, but not all quadruple points have reduced fundamental cycle. These rational surface singularities with reduced fundamental cycle are the subject of investigation of the second paper in this Habilitationsschrift. In this paper we were able to prove the above T^2–formula, and, furthermore, see enough from T^1 to give a more or less explicit description of the equations for the base spaces of such singularities, in some sense similar to the results of Arndt. To obtain these results, we were led by a certain heuristic method of degenerations to *tree singularities* that is explained in the third paper.
However, having the equations is a different matter than understanding the components. J. Christophersen was able, by looking very carefully at the obstruction map in the case of cyclic quotients, to give the right guess for the number of components in terms of the combinatorics of continued fractions representing zero. (J. Stevens [Stev2] then could prove this by combining the ideas of [K-S] and [Ch2] in a clever way). A similarly beautiful and simple result is not possible for general rational surfaces with reduced fundamental cycle, but there is some sort of substitute, the so–called *picture method*, a method under development by T. de Jong and the author. This method identifies the base space (up to a smooth factor) with a certain configuration space of points and curves in the plane. Informally, we have a collection of curves $\mathcal{C}_i$, and on each $\mathcal{C}_i$ a collection of points (or rather, a subscheme of finite colength) $\mathcal{P}_i \subset \mathcal{C}_i$. One now looks for those configurations where

$$(\mathcal{C}_i \cdot \mathcal{C}_j) \subset \mathcal{P}_i$$

holds for all i and j. For Pinkham's example one obtains as configuration three lines through one point, and on the intersection point a fat point of colength two on each of the lines.

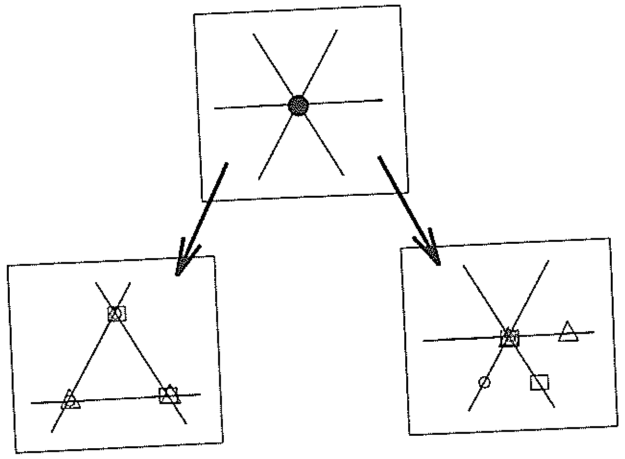

Figure 4: The Picture Method

The deformations of this configuration which satisfy the above condition come clearly in two types. Firstly, we can move the three lines apart to form a triangle. The points are now forced, on each line, to sit on the intersection points with the other two lines. Secondly, we can move one of the points on each line away from the intersection point. The other three points are now forced to stay behind, and furthermore, the three lines no longer can move apart without violating the condition. A little reflection will convince the reader now that again we have found the same space as in figure 2! This gives a nice *explanation* of **why** there are two components. Using the same method, one can *see* that the cones over the rational normal curve of higher degree have only one component. The main features of the theory are explained in the last part of the third paper in this Habilitationsschrift, with the title "Tree singularities: limits, series and stability". Tree singularities are certain non-isolated degenerations of series of rational surfaces with reduced fundamental cycle. Their importance can be understood by the fact that these singularities have a T^2 that is really the same as that of their series members. In fact more is true: the equations of the base space in such a series stabilizes after some point. This means that "high up in a series" the base space stays the same up to a smooth factor. This phenomenon was also clearly visible from the work of Arndt, and in our work on rational quadruple points. In this third paper one finds the first attempts to formulate this

important phenomenon which seems to be rather general.
The picture method has many promises, and we are just beginning to pick its fruits. In a forthcoming paper with T. de Jong [J-S6] we will describe many interesting applications to the component structure of rational surface singularities which seem to be totally out of reach of more traditional methods.

A deformation

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X} \\ \downarrow & & f\downarrow \\ \{0\} & \hookrightarrow & S \end{array}$$

can also be considered from another point of view: rather than to consider X as given, we might consider $\mathcal{X}$ as given, and study maps $\mathcal{X} \longrightarrow S$. The questions that arise are of a different nature. The simplest case is when we consider one–parameter deformations, say where S is a smooth curve germ. Then we are studying $\mathcal{X}$ together with a germ of a *function* f on it. The first case to consider here is the case where $\mathcal{X}$ is a smooth germ $\approx (\mathbf{C}^{n+1}, 0)$, [Mi]. The zero–fibre X has an isolated singularity precisely when the function f has an isolated critical point, which means that the partial derivatives $\partial f/\partial x_i$, $i = 0, \ldots, n$, have only 0 as common zero–set in some neigbourhood B of the origin. The Milnor fibre X_s has, in this situation, the homotopy type of a *bouquet of n-spheres*, and the number of spheres ($= \mu = \text{rank } H_n(X_s)$) can be computed algebraically as:

$$\mu = \dim_{\mathbf{C}}(\mathbf{C}\{x_0, \ldots, x_n\}/(\partial f/\partial x_0, \ldots, \partial f/\partial x_n).$$

We could also think of this number as the *multiplicity of the critical point of f*. The following *principle of conservation of number holds.* If we consider a one–parameter family of functions f_t with $f = f_0$, then we have:

$$\mu(f, 0) = \sum_{x \in B} \mu(f_t, x).$$

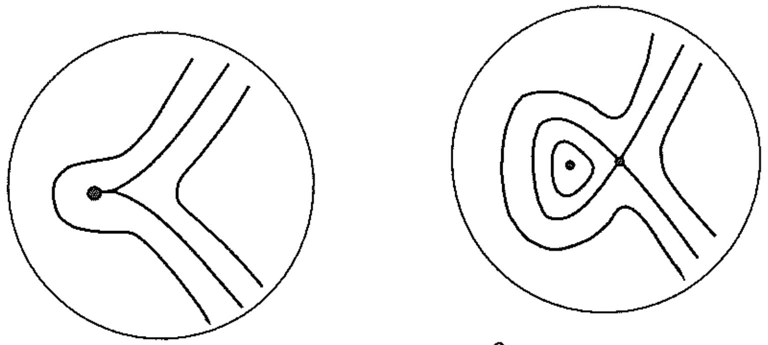

Figure 5: $\mu = 2$

In view of the topological meaning of the Milnor number, this constancy is clear, but algebraically, this is due to the fact that the partials of f form a regular sequence.
For functions on singular spaces things are not so straightforward. ON the topological side, there is a recent theorem of Siersma [Si4] that states that if $\mathcal{X}$ has an isolated singular point, then the Milnor fibre has homotopy type of a wedge of n–spheres, wedged with the Milnor fibre of a generic linear hypersurface section.
In [B-R] a general multiplicity of a critical point of a function $f : \mathcal{X} \longrightarrow \mathbf{C}$ was defined as:

$$\mu_{BR} = \dim_{\mathbf{C}}(\mathcal{O}_{\mathcal{X}}/(\Theta_{\mathcal{X}}(f))).$$

For this number to be preserved under perturbation in the above sense, the space $\mathcal{X}$ has to satisfy rather strong, and hard–to–verify conditions. What basically is needed is complete control over the vector fields on $\mathcal{X}$. Bruce and Roberts were able to show that these conditions were satisfied in the case that $\mathcal{X}$ is itself a quasi–homogeneous isolated complete intersection singularity and in the case that $\mathcal{X}$ is the quotient of a smooth space be a *finite* group. This last case, of course, is very important. In many applications one has a function f on a smooth space Y, *invariant* under some group acting on the space. Under these circumstances it is natural to consider the function to be defined on the quotient space Y/G. In the case that the group is *finite* this has been studied in some detail by [Ro], and in fact, by taking the invariant part of Jacobian algebra, one gets a formula for the number of critical G-orbits appearing in a generic invariant perturbation of f.
Somewhat surprisingly, this is no longer the case for infinite groups. Already for the simplest *infinite* groups, to wit $\mathbf{C}^*$, the invariant part of the Jacobian

ring is not the right thing to consider. The fifth paper of the Habilitationsschrift, written together with J. Montaldi, is concerned with the general case of $\mathbf{C}^*$–actions, and Milnor numbers for invariant functions. Our point of view is that it is more natural to consider *differential forms* on the quotient space, and generalise in an appropriate way the formula

$$\mu(f) = \dim_{\mathbf{C}} \Omega^{n+1}/df \wedge \Omega^n$$

which gives, in the smooth case, the same as before, but is, in the general case, different from μ_{BR}. Our number relates directly to the topology of the Milnor fibre, and is preserved under perturbation, as is to be expected in such a situation. Furthermore, the theory of the Gauß–Manin connection and the Brieskorn–lattice can be carried over to this situation (c.f. [S-S], [Bri4], [Ph]).
The fourth paper explains our motivation to look at the $\mathbf{C}^*$–case in so much detail: the theory of Hamiltonian systems near a resonant equilibrium point ([Dui]) give naturally rise to problems of this sort. In this paper the counting of *periodic orbits* of small period was solved in a special case using methods of algebraic geometry. As the paper was written for non–algebraic geometers, it contains an explanation of some standard techniques from this field of mathematics. The results of the fifth paper on $\mathbf{C}^*$–quotients can be applied to solve the problem of counting the periodic orbits in general, see [Mont]. Probably the techniques and ideas of the $\mathbf{C}^*$–paper can be extended to include more general quotients, and functions on affine toric manifolds. These extensions are currently under investigation, and seem to have very promising applications in theoretical physics.

I have mentioned before that the general theory of the cotangent complex was not of very much use for the explicit understanding of base spaces. The sixth paper however shows that one can prove some nice results using this general machinery. The base space $\mathcal{B}$ of the semi–universal deformation of a space curve X is smooth. Also, the general fibre X_s will be smooth, but inside $\mathcal{B}$ we find a hypersurface of points where the fibre X_s is singular, the *discriminant* Δ. As I pointed out earlier, the study of space curves is important for understanding normal surface singularities, and it is, in particular, the stratification of the discriminant which is relevant. We prove that this hypersurface is a *free divisor*, what means that the module of vector fields tangent to the discriminant is a free module. The concept of a free divisor was introduced by K. Saito [Sa], who also proved that the discriminant of the semi–universal deformation of isolated complete intersection singularities is an example of such a free divisor. Discriminants in general are very singular objects, but the freeness is a property that they have in common with *smooth* hypersurfaces. As freeness means complete control over the vector fields, one can effectively study functions $\phi : \Delta \longrightarrow \mathbf{C}$ on free divisors. In ([D-M]) it was shown that the so-called singular Milnor fibre of a section $S \hookrightarrow \Delta$ with a smooth space has the homotopy type of a wedge of spheres, the number of which can be calculated algebraically. Recently, J. Damon has generalized these ideas, and proposes the category of *almost free divisors* as a proper framework to formulate results of this type ([Da]).

In the seventh paper the construction of a *quintic hypersurface in* $\mathbf{P}^4$ with 130 nodes as singularities is described. As this type of result is slightly out of line with the other papers, let me try to put this in some perspective. If a hypersurface X_s is the nearby fibre of an isolated hypersurface singularity X, then on has:

$$\mu(X) \geq \sum_{x \in X_s} \mu((X_s, x))$$

so the Milnor number is semi–continuous under deformation. The *spectrum of* X can be thought of as a *refinement* of the Milnor number and was introduced by J. Steenbrink in [Stee1], see also [Stee4]. It consists of a set of μ rational numbers, which are *logarithms* of eigenvalues of the monodromy, where the integer part is determined by the Hodge filtration on the cohomology of the Milnor fibre. It was Arnol'd who conjectured a semi-continuity property for the spectrum, and Varchenko [Va] who proved it for quasi–homogeneous singularities. Finally, J. Steenbrink proved the semi-continuity property in general, stating that the number of spectrum numbers in any interval of the form $(a, a+1]$ of X is bigger than or equal to the number of spectrum numbers of all the singularities of X_s in the same interval.
Varchenko also gave a beautiful application of the semi–continuity of the spectrum to a global problem of singularity theory, which we will describe now. In general, the maximum number $N_n(d)$ of ordinary double points a hypersurface of degree d in $\mathbf{P}^n$ can have is unknown. As the complement of a general hyperplane section of any hypersurface can be seen as a deformation of the affine cone over this section, the semi–continuity of the spectrum can be used to get an *upperbound* for $N_n(d)$, and it turns out to be a very good

one in general. Let us see what is known about $N_n(d)$ for low values for n and d. It is easy to see that in $\mathbf{P}^2$ a curve of degree d with maximal number of double points must be a union of lines in general position. The number of ordinary double points is then clearly $\binom{d}{2} = N_2(d)$.

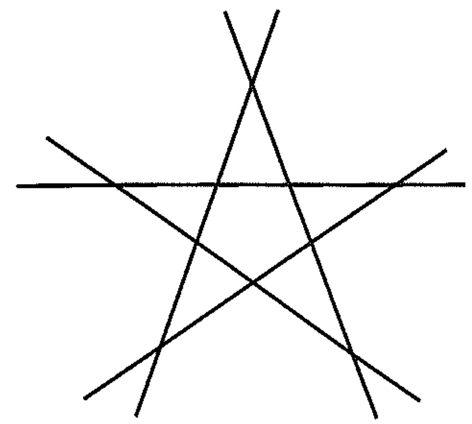

Figure 6: $N_2(5) = 10$

For surfaces in $\mathbf{P}^3$ one has $N_3(2) = 1$ (the ordinary cone), $N_3(3) = 4$ (the Cayley Cubic, see [Cay]), $N_3(4) = 16$ (a Kummer Quartic, see [Hu]), $N_3(5) = 31$ (a Togliatti Quintic, see [To]). The precise value of $N_3(d)$ for higher values of d seems to be unknown.
For solids in $\mathbf{P}^4$ one has $N_4(2) = 1$, $N_4(3) = 10$ (the Segre Cubic, [Se]), $N_4(4) = 45$ (the Burkhardt Quartic, [Bur]). These last two varieties have many remarkable properties, and their geometry was studied very thoroughly by the classical geometers. Nevertheless, new properties of these varieties were discovered recently ([H-W]).
For all these cases Varchenko's upperbound is sharp. The value of $N_4(5)$ is unknown, but it follows from Varchenko's spectral bound that $N_4(5) \leq 135$. The quintic with equation

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5x_0x_1x_2x_3x_4 = 0$$

is called *Schoen's Quintic* and has 125 ordinary double points, [Scho]. In [Hi] Hirzebruch managed to construct a quintic with 126 nodes. He took as equation (the homogenisation of)

$$F(x, y) - F(z, t) = 0$$

where $F(x,y) = 0$ is the equation of a regular pentagon, as in figure 5. The construction of the quintic with 130 nodes is surprisingly simple, as the reader will find out by reading the last paper. At this moment of writing, it is the world record, and it is very well concievable that $N_4(5) = 130$. Quintics in $\mathbf{P}^4$ are the simplest examples of *Calabi-Yau manifolds*, which are very much in the center of current interest in high energy physics and algebraic geometry, see [Can], [Mor].

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